**Math 150, *Lecture Notes- Bonds*** **Name____________**

***Section 3.9*** *Differentials*

## Tangent Line Approximations

To begin, consider a function $f$ that is differentiable at $c$. The equation for the tangent line at the point $(c, f(c))$ is given by

$$y - f(c) = f'(c)(x - c)$$

$$\boxed{y = f(c) + f'(c)(x - c)}$$

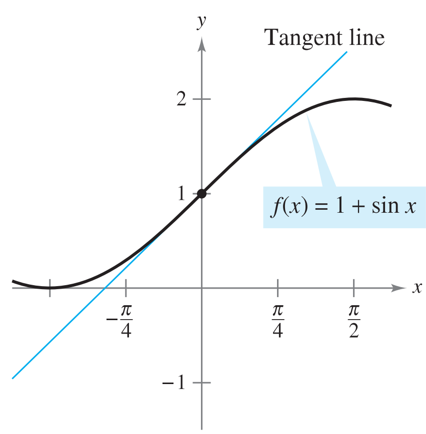
and is called the **tangent line approximation** (or **linear approximation**) **of *f* at *c*.** Because $c$ is a constant, $y$ is a linear function of $x$. Moreover, by restricting the values of $x$ to those sufficiently close to $c$, the values of $y$ can be used as approximations (to any desired degree of accuracy) of the values of the function $f$. In other words, as $x \to c$, the limit of $y$ is $f(c)$.

**Ex.1** Using a Tangent Line Approximation

Find the tangent line approximation of

$$f(x) = 1 + \sin x$$

at the point $(0, 1)$. Then use a table to compare the $y$-values of the linear function with those of $f(x)$ on an open interval containing $x = 0$.




The tangent line approximation of $f$ at the point $(0, 1)$
**Figure 3.65**

| $x$ | $-0.5$ | $-0.1$ | $-0.01$ | $0$ | $0.01$ | $0.1$ | $0.5$ |
|---|---|---|---|---|---|---|---|
| $f(x) = 1 + \sin x$ | 0.521 | 0.9002 | 0.9900002 | 1 | 1.0099998 | 1.0998 | 1.479 |
| $y = 1 + x$ | 0.5 | 0.9 | 0.99 | 1 | 1.01 | 1.1 | 1.5 |

## Differentials

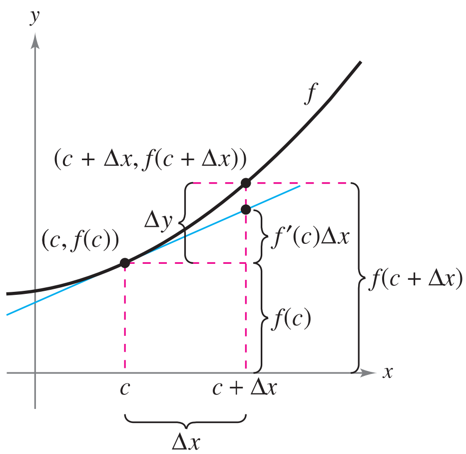
When the tangent line to the graph of $f$ at the point $(c, f(c))$

$$y = f(c) + f'(c)(x - c) \qquad \text{Tangent line at } (c, f(c))$$

is used as an approximation of the graph of $f$, the quantity $x - c$ is called the change in $x$, and is denoted by $\Delta x$, as shown in Figure 3.66. When $\Delta x$ is small, the change in $y$ (denoted by $\Delta y$) can be approximated as shown.

$$\Delta y = f(c + \Delta x) - f(c) \qquad \text{Actual change in } y$$
$$\approx f'(c)\Delta x \qquad \text{Approximate change in } y$$

For such an approximation, the quantity $\Delta x$ is traditionally denoted by $dx$, and is called the **differential of *x*.** The expression $f'(x)\,dx$ is denoted by $dy$, and is called the **differential of *y*.**




When $\Delta x$ is small, $\Delta y = f(c + \Delta x) - f(c)$ is approximated by $f'(c)\Delta x$.
**Figure 3.66**

In many types of applications, the differential of $y$ can be used as an approximation of the change in $y$. That is,

$$\Delta y \approx dy \qquad \text{or} \qquad \Delta y \approx f'(x)dx.$$

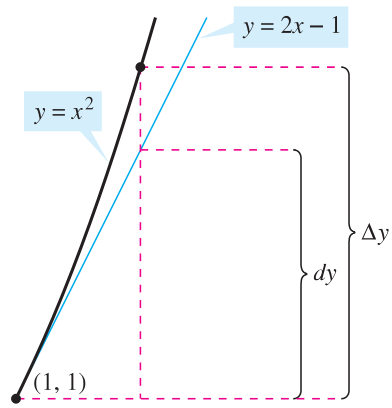
### Definition of Differentials

Let $y = f(x)$ represent a function that is differentiable on an open interval containing $x$. The **differential of *x*** (denoted by $dx$) is any nonzero real number. The **differential of *y*** (denoted by $dy$) is

$$dy = f'(x)\,dx.$$

## Ex.2 Comparing $\Delta y$ and $dy$

Let $y = x^2$. Find $dy$ when $x = 1$ and $dx = 0.01$. Compare this value with $\Delta y$ for $x = 1$ and $\Delta x = 0.01$.




The change in $y$, $\Delta y$, is approximated by the differential of $y$, $dy$.
**Figure 3.67**

Figure 3.67 shows the geometric comparison of $dy$ and $\Delta y$. Try comparing other values of $dy$ and $\Delta y$. You will see that the values become closer to each other as $dx$ (or $\Delta x$) approaches 0.

In Example 2, the tangent line to the graph of $f(x) = x^2$ at $x = 1$ is

$$y = 2x - 1 \quad \text{or} \quad g(x) = 2x - 1.$$ Tangent line to the graph of $f$ at $x = 1$.

For $x$-values near 1, this line is close to the graph of $f$, as shown in Figure 3.67. For instance,

$$f(1.01) = 1.01^2 = 1.0201 \quad \text{and} \quad g(1.01) = 2(1.01) - 1 = 1.02.$$

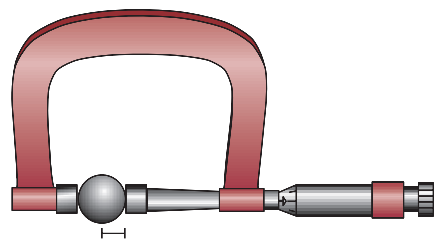
## Error Propagation

Physicists and engineers tend to make liberal use of the approximation of $\Delta y$ by $dy$. One way this occurs in practice is in the estimation of errors propagated by physical measuring devices. For example, if you let $x$ represent the measured value of a variable and let $x + \Delta x$ represent the exact value, then $\Delta x$ is the *error in measurement*. Finally, if the measured value $x$ is used to compute another value $f(x)$, the difference between $f(x + \Delta x)$ and $f(x)$ is the **propagated error.**

$$\underbrace{f(x + \overbrace{\Delta x}^{\text{Measurement error}})}_{\text{Exact value}} - \underbrace{f(x)}_{\text{Measured value}} = \overbrace{\Delta y}^{\text{Propagated error}}$$

**Ex.3** Estimation of Error

The measured radius of a ball bearing is 0.7 inch, as shown in Figure 3.68. If the measurement is correct to within 0.01 inch, estimate the propagated error in the volume $V$ of the ball bearing.




Ball bearing with measured radius that is correct to within 0.01 inch.
**Figure 3.68**

Would you say that the propagated error in Example 3 is large or small? The answer is best given in *relative* terms by comparing $dV$ with $V$. The ratio

$$\frac{dV}{V} =$$

Ratio of $dV$ to $V$

$$\approx \pm 0.0429$$

is called the **relative error.** The corresponding **percent error** is approximately 4.29%.

## Calculating Differentials

Each of the differentiation rules that you studied in Chapter 2 can be written in **differential form.** For example, suppose $u$ and $v$ are differentiable functions of $x$. By the definition of differentials, you have

$$du = u'\,dx \quad \text{and} \quad dv = v'\,dx.$$

So, you can write the differential form of the Product Rule as shown below.

$$\begin{aligned}
d[uv] &= \frac{d}{dx}[uv]\,dx && \text{Differential of } uv \\
&= [uv' + vu']\,dx && \text{Product Rule} \\
&= uv'\,dx + vu'\,dx \\
&= u\,dv + v\,du
\end{aligned}$$

### Differential Formulas

Let $u$ and $v$ be differentiable functions of $x$.

***Constant multiple:*** $d[cu] = c\,du$

***Sum or difference:*** $d[u \pm v] = du \pm dv$

***Product:*** $d[uv] = u\,dv + v\,du$

***Quotient:*** $d\left[\frac{u}{v}\right] = \frac{v\,du - u\,dv}{v^2}$

**Ex.4** Finding Differentials

| | *Function* | *Derivative* | *Differential* |
|---|---|---|---|
| **a.** | $y = x^2$ | $\frac{dy}{dx} = 2x$ | $dy = 2x\,dx$ |
| **b.** | $y = 2\sin x$ | $\frac{dy}{dx} = 2\cos x$ | $dy = 2\cos x\,dx$ |
| **c.** | $y = x\cos x$ | $\frac{dy}{dx} = -x\sin x + \cos x$ | $dy = (-x\sin x + \cos x)\,dx$ |
| **d.** | $y = \frac{1}{x}$ | $\frac{dy}{dx} = -\frac{1}{x^2}$ | $dy = -\frac{dx}{x^2}$ |

The notation in Example 4 is called the **Leibniz notation** for derivatives and differentials, named after the German mathematician Gottfried Wilhelm Leibniz. The beauty of this notation is that it provides an easy way to remember several important calculus formulas by making it seem as though the formulas were derived from algebraic manipulations of differentials. For instance, in Leibniz notation, the *Chain Rule*

$$\frac{dy}{dx} = \frac{dy}{du}\frac{du}{dx}$$

would appear to be true because the $du$'s divide out. Even though this reasoning is *incorrect*, the notation does help one remember the Chain Rule.

**Ex.5** Find the Differential

Find $dy$ if $y = \sqrt{9 - x^2}$ .

**Ex.6** Find the Differential

Find $dy$ if $y = x\cos(2x)$.

Differentials can be used to approximate function values. To do this for the function given by $y = f(x)$, use the formula

$$f(x + \Delta x) \approx f(x) + dy = f(x) + f'(x)\,dx$$

which is derived from the approximation $\Delta y = f(x + \Delta x) - f(x) \approx dy$. The key to using this formula is to choose a value for $x$ that makes the calculations easier, as shown in Example 7. (This formula is equivalent to the tangent line approximation given earlier in this section.)

## Ex.7 Approximating Function Values

Use differentials to approximate $\sqrt{16.5}$.

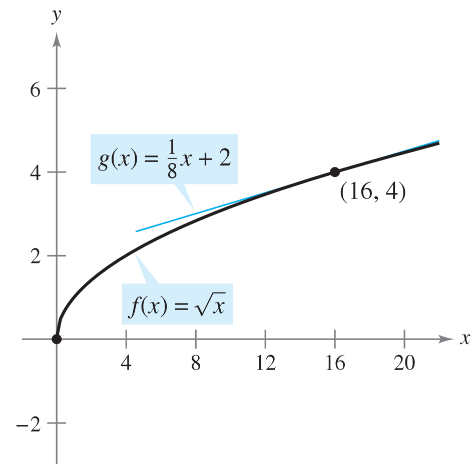



**Figure 3.69**

The tangent line approximation to $f(x) = \sqrt{x}$ at $x = 16$ is the line $g(x) = \frac{1}{8}x + 2$. For $x$-values near 16, the graphs of $f$ and $g$ are close together, as shown in Figure 3.69. For instance,

$$f(16.5) = \sqrt{16.5} \approx 4.0620 \qquad \text{and} \qquad g(16.5) = \frac{1}{8}(16.5) + 2 = 4.0625.$$

In fact, if you use a graphing utility to zoom in near the point of tangency $(16, 4)$, you will see that the two graphs appear to coincide. Notice also that as you move farther away from the point of tangency, the linear approximation becomes less accurate.